

STATISTICAL EXTENSION OF CLASSICAL TAUBERIAN THEOREMS IN THE CASE OF LOGARITHMIC SUMMABILITY OF LOCALLY INTEGRABLE FUNCTIONS ON $[1, \infty)$

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ABSTRACT. Let $s : [1, \infty) \rightarrow \mathbb{C}$ be a locally integrable function in Lebesgue's sense. The logarithmic (also called harmonic) mean of the function s is defined by

$$\tau(t) := \frac{1}{\log t} \int_1^t \frac{s(x)}{x} dx, \quad t > 1,$$

where the logarithm is to base e . Besides the ordinary limit $\lim_{x \rightarrow \infty} s(x)$, we also use the notion of the so-called statistical limit of s at ∞ , in notation: $\text{st-lim}_{x \rightarrow \infty} s(x) = \ell$, by which we mean that for every $\varepsilon > 0$,

$$\lim_{b \rightarrow \infty} \frac{1}{b} \left| \left\{ x \in (1, b) : |s(x) - \ell| > \varepsilon \right\} \right| = 0.$$

We also use the ordinary limit $\lim_{t \rightarrow \infty} \tau(t)$ as well as the statistical limit $\text{st-lim}_{t \rightarrow \infty} \tau(t)$.

We will prove the following Tauberian theorem: Suppose that the real-valued function s is slowly decreasing or the complex-valued s is slowly oscillating. If the statistical limit $\text{st-lim}_{t \rightarrow \infty} \tau(t) = \ell$ exists, then the ordinary limit $\lim_{x \rightarrow \infty} s(x) = \ell$ also exists.

1. INTRODUCTION

We consider real- and complex-valued functions which are measurable (in Lebesgue's sense) on some interval (a, ∞) , where $a \geq 0$. We recall (see in [6]) that a function s has *statistical limit at ∞* if there exists a number ℓ such that for every $\varepsilon > 0$,

$$(1.1) \quad \lim_{b \rightarrow \infty} \frac{1}{b-a} \left| \left\{ x \in (a, b) : |s(x) - \ell| > \varepsilon \right\} \right| = 0,$$

where by $|\{\cdot\}|$ we denote the Lebesgue measure of the set indicated in $\{\cdot\}$. If this is the case, we write

$$(1.2) \quad \text{st-lim}_{x \rightarrow \infty} s(x) = \ell.$$

Clearly, the statistical limit ℓ is uniquely determined if it exists. The particular choice of the left endpoint a in the definition domain of the function s is indifferent in (1.1). Furthermore, if the ordinary limit

$$(1.3) \quad \lim_{x \rightarrow \infty} s(x) = \ell$$

exists, then (1.2) also exists. But the converse implication is not true in general.

1991 *Mathematics Subject Classification.* Primary 40C10, Secondary 40E05, 40G05.

Key words and phrases. Statistical limit of a measurable function at ∞ , logarithmic summability $(L, 1)$ of a locally integrable function, slow decrease and slow oscillation with respect to summability $(L, 1)$, nondiscrete version of a Vijayaraghavan lemma, Landau type one-sided and Hardy type two-sided Tauberian conditions.

We note that the notion of the statistical limit of a measurable function at ∞ is the nondiscrete version of the notion of the statistical convergence of a sequence of real or complex numbers, which was introduced by Fast [1].

If the function $s : [1, \infty) \rightarrow \mathbb{C}$ is integrable in Lebesgue's sense in every bounded interval $(1, t)$, $t > 1$, in symbols: $f \in L^1_{\text{loc}}[1, \infty)$, then its logarithmic (also called harmonic) mean $\tau(t)$ of order 1 is defined by

$$(1.4) \quad \tau(t) := \frac{1}{\log t} \int_1^t \frac{s(x)}{x} dx, \quad t > 1,$$

where the logarithm is to the natural base e . The function s is said to be *logarithmic summable at ∞* , or briefly *summable $(L, 1)$* , if the finite limit

$$(1.5) \quad \lim_{t \rightarrow \infty} \tau(t) = \ell$$

exists. It is easy to check that if the ordinary limit (1.3) exists, then (1.5) also exists with the same ℓ .

The converse implication (1.5) \Rightarrow (1.3) is usually false. However, if we subject the function s to an appropriate additional condition, then the implication (1.5) \Rightarrow (1.3) does hold. Such conditions are called ‘Tauberian’ ones, and the theorems involving such conditions are also called ‘Tauberian’ ones; after A. Tauber [10], who first proved one of the simplest of them.

We recall (see in [8]), that a function $s : [1, \infty) \rightarrow \mathbb{R}$ is said to be *slowly decreasing* with respect to logarithmic summability, or briefly: summability $(L, 1)$, if

$$(1.6) \quad \lim_{\lambda \rightarrow 1^+} \liminf_{x \rightarrow \infty} \inf_{\log x < \log t \leq \lambda \log x} (s(t) - s(x)) \geq 0.$$

Since the auxiliary function

$$a(\lambda) := \liminf_{x \rightarrow \infty} \inf_{\log x < \log t \leq \lambda \log x} (s(t) - s(x)), \quad \lambda > 1,$$

is clearly decreasing in λ on the interval $(1, \infty)$, the term ‘ $\lim_{\lambda \rightarrow 1^+}$ ’ in (1.6) can be equivalently replaced by ‘ $\sup_{\lambda > 1}$ ’.

We observe that the conditions

$$\log x < \log t \leq \lambda \log x \quad \text{and} \quad x < t \leq x^\lambda, \quad x > 1,$$

are equivalent. In the sequel, we will use the second one of them. It is easy to check that condition (1.6) is satisfied if and only if for every $\varepsilon > 0$ there exist $x_0 = x_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$, the latter one is as close to 1 as we want, such that

$$(1.7) \quad s(t) - s(x) \geq -\varepsilon \quad \text{whenever} \quad x_0 \leq x < t \leq x^\lambda.$$

Historically, the term ‘slow decrease’ was introduced by Schmidt [9] (see also in [3, p. 124]), in the case of the summability $(C, 1)$ of sequences of real numbers.

We note that the symmetric counterpart of the notion of slow decrease is the following one: a real-valued function s is said to be *slowly increasing* with respect to summability $(L, 1)$ if

$$\lim_{\lambda \rightarrow 1^+} \limsup_{x \rightarrow \infty} \sup_{x < t \leq x^\lambda} (s(t) - s(x)) \leq 0.$$

Clearly, a function s is slowly increasing if and only if the function $(-s)$ is slowly decreasing.

We recall (see in [8]), that a function $s : [1, \infty) \rightarrow \mathbb{C}$ is said to be *slowly oscillating* with respect to summability $(L, 1)$ if

$$(1.8) \quad \lim_{\lambda \rightarrow 1^+} \limsup_{x \rightarrow \infty} \sup_{x < t \leq x^\lambda} |s(t) - s(x)| = 0.$$

Again, the term ‘ $\lim_{\lambda \rightarrow 1^+}$ ’ in (1.8) can be equivalently replaced by ‘ $\inf_{\lambda > 1}$ ’.

Analogously to (1.7), condition (1.8) is satisfied if and only if for every $\varepsilon > 0$ there exist $x_0 = x_0(\varepsilon) > 1$ and $\lambda = \lambda(\varepsilon) > 1$, the latter one is as close as to 1 as we want, such that

$$(1.9) \quad |s(t) - s(x)| \leq \varepsilon \quad \text{whenever} \quad x_0 \leq x < t \leq x^\lambda.$$

It is easy to see that a real-valued function s is slowly oscillating if and only if s is both slowly decreasing and slowly increasing.

Historically, the term ‘slow oscillation’ was introduced by Hardy [2] (see also in [3, p. 124]), in the case of the summability $(C, 1)$ of sequences of numbers.

We note that in the special case when

$$s(x) := \int_1^x f(u) du, \quad x \geq 1,$$

where $f \in L^1_{\text{loc}}[1, \infty)$, one can easily get sufficient conditions for the fulfillment of (1.7) and (1.9), respectively. If f is a real-valued function such that

$$(1.10) \quad u(\log u)f(u) \geq -C \quad \text{at almost every} \quad u > x_0,$$

where $C > 0$ and $x_0 \geq 1$ are constants, then the function s is slowly decreasing with respect to summability $(L, 1)$. Furthermore, if f is a complex-valued function such that

$$(1.11) \quad (\log u)|f(u)| \leq C \quad \text{at almost every} \quad u > x_0,$$

where $C > 0$ and $x_0 \geq 1$ are constants, then the function s is slowly oscillating with respect to summability $(L, 1)$.

Condition (1.10) is called a one-sided Tauberian condition, while (1.11) is called a two-sided Tauberian condition. These terms go back to Landau [5] with respect to summability $(C, 1)$ of sequences of real numbers; and to Hardy [2] (see also in [3, p. 124]) with respect to summability $(C, 1)$ of sequences of real or complex numbers.

The following two classical Tauberian theorems were proved in [8, Corollaries 1 and 2].

Theorem A. *If a function $s : [1, \infty) \rightarrow \mathbb{R}$ is slowly decreasing with respect to summability $(L, 1)$, then the implication $(1.5) \Rightarrow (1.3)$ holds true.*

Theorem B. *If a function $s : [1, \infty) \rightarrow \mathbb{C}$ is slowly oscillating with respect to summability $(L, 1)$, then the implication $(1.5) \Rightarrow (1.3)$ holds true.*

We note that in the case of sequences of real numbers, a theorem analogous to Theorem A was proved by Kwee [4, Lemma 3].

2. NEW RESULTS

First, we prove that if a measurable function s is slowly decreasing or oscillating with respect to summability $(L, 1)$, then the existence of the statistical limit ℓ of s at ∞ implies the existence of the ordinary limit of s at ∞ to the same limit ℓ .

Theorem 1. *If a real-valued function $s : [1, \infty) \rightarrow \mathbb{R}$ is measurable and slowly decreasing with respect to summability $(L, 1)$, then the implication $(1.2) \Rightarrow (1.3)$ holds true.*

Theorem 2. *If a complex-valued function $s : [1, \infty) \rightarrow \mathbb{C}$ is measurable and slowly oscillating with respect to summability $(L, 1)$, then the implication $(1.2) \Rightarrow (1.3)$ holds true.*

The next two theorems are the main results of the present paper. They state that under the Tauberian condition of slow decrease or slow oscillation, respectively, (1.3) follows from the existence of the even weaker limit

$$(2.1) \quad \text{st-}\lim_{t \rightarrow \infty} \tau(t) = \ell.$$

Theorem 3. *If $s \in L^1_{\text{loc}}[1, \infty)$ is a real-valued function and slowly decreasing with respect to summability $(L, 1)$, then the implication $(2.1) \Rightarrow (1.3)$ holds true.*

Theorem 4. *If $s \in L^1_{\text{loc}}[1, \infty)$ is a complex-valued function and slowly oscillating with respect to summability $(L, 1)$, then the implication $(2.1) \Rightarrow (1.3)$ holds true.*

We note that analogous theorems were proved in [7] for sequences of real and complex numbers, respectively. However, the method of the proof in the present paper is more straightforward than that in [7]. As a result, the present proofs are essentially shorter and more transparent than those in [7].

3. AUXILIARY RESULTS

Our Lemma 1 is analogous to the famous Vijayaraghavan lemma (see in [11] and also in [3, Theorem 239 on p. 307]), which relates to the slow decrease with respect to summability $(C, 1)$ in the case of sequences of real numbers. Our Lemma 1 relates to slow decrease with respect to summability $(L, 1)$ in the case of real-valued functions.

Lemma 1. *If a function $s : [1, \infty) \rightarrow \mathbb{R}$ is such that the condition (1.7) is satisfied only for $\varepsilon := 1$, where $x_0 > 1$ and $\lambda > 1$, then there exists a constant $B_1 > 0$ such that*

$$(3.1) \quad s(t) - s(x) \geq -B_1 \log \left(\frac{\log t}{\log x} \right) \quad \text{whenever} \quad x_0 \leq x < t^{1/\lambda}.$$

Proof. Given $x_0 \leq x < t^{1/\lambda}$, we form the decreasing sequence

$$(3.2) \quad t_0 := t, \quad t_p := t_{p-1}^{1/\lambda}, \quad p = 1, 2, \dots, q+1,$$

where q is defined by the condition

$$(3.3) \quad t_{q+1} \leq x < t_q.$$

By (1.7) and (3.3), we estimate as follows:

$$(3.4) \quad s(t) - s(x) = \sum_{p=1}^q (s(t_{p-1}) - s(t_p)) + (s(t_q) - s(x)) \geq -q - 1.$$

It is clear that

$$(3.5) \quad \frac{1}{\lambda^q} \log t > \log x, \quad \text{or equivalently} \quad q < \frac{1}{\log \lambda} \log \left(\frac{\log t}{\log x} \right).$$

Combining (3.4) and (3.5) gives

$$(3.6) \quad s(t) - s(x) \geq -1 - \frac{1}{\log \lambda} \log \left(\frac{\log t}{\log x} \right) \quad \text{whenever } x_0 \leq x < t^{1/\lambda}.$$

Since it follows from $x < t^{1/\lambda}$ that

$$(3.7) \quad \log \lambda < \log \left(\frac{\log t}{\log x} \right),$$

we conclude from (3.6) that (3.1) holds with $B_1 := 2/\log \lambda$. \square

Our Lemma 2 is the counterpart of Lemma 1 in the case of complex-valued functions.

Lemma 2. *If a function $s : [1, \infty) \rightarrow \mathbb{C}$ is such that the condition (1.9) is satisfied only for $\varepsilon := 1$, where $x_0 > 1$ and $\lambda > 1$, then with $B_1 := 2/\log \lambda$ we have*

$$(3.8) \quad |s(t) - s(x)| \leq B_1 \log \left(\frac{\log t}{\log x} \right) \quad \text{whenever } x_0 \leq x < t^{1/\lambda}.$$

Proof. It goes along the same lines as the proof of Lemma 1. For given $x_0 \leq x < t^{1/\lambda}$, we define t_0, t_1, \dots, t_{q+1} by (3.2) and (3.3). Using (1.9) and (3.4) gives

$$(3.9) \quad |s(t) - s(x)| \leq \sum_{p=1}^q |s(t_{p-1} - s(t_p))| + |s(t_q) - s(x)| \leq q + 1.$$

Combining (3.5) and (3.9) we obtain

$$|s(t) - s(x)| \leq 1 + \frac{1}{\log \lambda} \log \left(\frac{\log t}{\log x} \right) \quad \text{whenever } x_0 \leq x < t^{1/\lambda}$$

(c.f. (3.6)). Taking into account (3.7), hence (3.8) follows with the same constant $B_1 := 2/\log \lambda$ as in Lemma 1. \square

The next two lemmas are corollaries of Lemmas 1 and 2, respectively.

Lemma 3. *Under the assumptions of Lemma 1, there exists a constant $B_2 > 0$ such that*

$$(3.10) \quad \frac{1}{\log t} \int_{x_0}^t \frac{s(t) - s(x)}{x} dx \geq -B_2 \quad \text{whenever } t > x_0^\lambda.$$

Proof. Without loss of generality, we may assume that $x_0 > e$. By (1.7) with $\varepsilon := 1$ and (3.1), we estimate as follows:

$$(3.11) \quad \begin{aligned} \int_{x_0}^t \frac{s(t) - s(x)}{x} dx &= \left\{ \int_{x_0}^{t^{1/\lambda}} + \int_{t^{1/\lambda}}^t \right\} \frac{s(t) - s(x)}{x} dx \\ &\geq -B_1 \int_{x_0}^{t^{1/\lambda}} \frac{1}{x} \log \left(\frac{\log t}{\log x} \right) dx - \int_{t^{1/\lambda}}^t \frac{dx}{x} \\ &\geq -B_1 (\log \log t) \int_{x_0}^{t^{1/\lambda}} \frac{dx}{x} + B_1 \int_{x_0}^{t^{1/\lambda}} \frac{\log \log x}{x} dx - \frac{\lambda - 1}{\lambda} \log t. \end{aligned}$$

Integration by parts gives

$$\begin{aligned}
 (3.12) \quad \int_{x_0}^{t^{1/\lambda}} \frac{\log \log x}{x} dx &= \left[(\log \log x) \log x \right]_{x_0}^{t^{1/\lambda}} - \int_{x_0}^{t^{1/\lambda}} \frac{dx}{x} \\
 &= (\log \log(t^{1/\lambda})) \log t^{1/\lambda} - (\log \log x_0) \log x_0 - \log t^{1/\lambda} + \log x_0 \\
 &= \frac{(\log \log t) \log t}{\lambda} - \frac{\log \lambda}{\lambda} \log t - (\log \log x_0) \log x_0 - \log t^{1/\lambda} + \log x_0.
 \end{aligned}$$

Returning to (3.11), we obtain

$$\begin{aligned}
 (3.13) \quad \int_{x_0}^t \frac{s(t) - s(x)}{x} dx &\geq -B_1 \frac{\log \lambda}{\lambda} \log t - B_1 (\log \log x_0) \log x_0 - B_1 \frac{\log t}{\lambda} + B_1 \log x_0 \\
 &\geq -B_1 (\log t) \left(\frac{\log \lambda}{\lambda} + \frac{(\log \log x_0) \log x_0}{\log t} + \frac{1}{\lambda} \right) \\
 &\geq -B_1 (\log t) \left(\frac{\log \lambda}{\lambda} + \frac{(\log \log x_0)}{\lambda} + \frac{1}{\lambda} \right),
 \end{aligned}$$

where we took into account that $(\log x_0)/(\log t) < 1/\lambda$. Now, the last expression on the right-hand side of (3.13) proves (3.11) with the constant

$$(3.14) \quad B_2 := \frac{B_1}{\lambda} (\log \lambda + \log \log x_0 + 1).$$

□

The counterpart of Lemma 3 in the case of complex-valued functions reads as follows.

Lemma 4. *Under the assumptions of Lemma 2, there exists a constant $B_2 > 0$ such that*

$$(3.15) \quad \frac{1}{\log t} \int_{x_0}^t \frac{|s(t) - s(x)|}{x} dx \leq B_2 \quad \text{whenever } t > x_0^\lambda.$$

Proof. It goes along analogous lines as the proof of Lemma 3. By (1.9) with $\varepsilon := 1$ and (3.8), we estimate as follows.

$$(3.16) \quad \int_{x_0}^t \frac{|s(t) - s(x)|}{x} dx \leq B_1 \int_{x_0}^{t^{1/\lambda}} \frac{1}{x} \log \left(\frac{\log t}{\log x} \right) dx + \int_{t^{1/\lambda}}^t \frac{dx}{x}$$

(cf. (3.11)). Combining (3.12) and 3.16 we obtain

$$(3.17) \quad \int_{x_0}^t \frac{|s(t) - s(x)|}{x} dx \leq B_1 (\log t) \left(\frac{\log \lambda}{\lambda} + \frac{(\log \log x_0)}{\lambda} + \frac{1}{\lambda} \right).$$

Thus, we have proved (3.15) with the same constant B_2 as given in (3.14). □

4. PROOFS OF THEOREMS 1–4

Proof of Theorem 1. Let $\varepsilon > 0$, $x_0 > 1$ and $\lambda > 1$ be arbitrarily given. By assumption, the statistical limit ℓ of the function s exists at ∞ . Thus, by (1.1) with $a := 1$, there exists $b_1 \geq x_0$ such that

$$|s(b_1) - \ell| \leq \varepsilon.$$

We distinguish between two cases:

(i) There exists some $b_2 \in (b_1^{\sqrt{\lambda}}, b_1^\lambda)$ such that

$$(4.1) \quad |s(b_2) - \ell| \leq \varepsilon;$$

(ii) There is no such b_2 ; that is, we have

$$|s(t) - \ell| > \varepsilon \quad \text{for every } t \in (b_1^{\sqrt{\lambda}}, b_1^\lambda).$$

In the latter case, we choose some $b_2 \geq b_1^\lambda$ for which (4.1) is satisfied. Due to (1.1), such b_2 certainly exists.

We repeat the previous step by starting with b_2 , and so on. As a result, we obtain an increasing sequence $(b_n : n = 1, 2, \dots)$ of numbers such that

$$(4.2) \quad |s(b_n) - \ell| \leq \varepsilon, \quad n = 1, 2, \dots$$

We claim that the case when

$$(4.3) \quad |s(t) - \ell| > \varepsilon, \quad \text{for every } t \in (b_n^{\sqrt{\lambda}}, b_n^\lambda)$$

cannot occur for infinitely many values of n . Otherwise, for infinitely many n we would have

$$\frac{1}{b_n} \left| \{t \in (1, b_n) : |s(t) - \ell| > \varepsilon\} \right| = b_n^{\lambda-1} - b_n^{\sqrt{\lambda}-1} > b_1^{\lambda-1} - b_1^{\sqrt{\lambda}-1} > 0,$$

and this contradicts (1.1). Consequently, inequality (4.3) can occur only for finitely many values of n . Denote by n_0 the largest value of n for which (4.3) holds (perhaps $n_0 = 0$ in the case when (4.3) does not occur at all). Consequently, we have

$$(4.4) \quad b_{n+1} < b_n^\lambda \quad \text{for } n = n_0 + 1, n_0 + 2, \dots$$

On the other hand, by definition we also have

$$b_{n+1} > b_n^{\sqrt{\lambda}} \quad \text{for } n = n_0 + 1, n_0 + 2, \dots,$$

whence it follows that

$$\lim_{n \rightarrow \infty} b_n = \infty.$$

By the condition (1.7) of slow decrease, we have

$$(4.5) \quad s(t) - s(b_n) \geq -\varepsilon \quad \text{whenever } x_0 \leq b_n < t \leq b_n^\lambda, \quad n > n_0.$$

Now, let $t \in (b_n, b_{n+1}]$ for some $n > n_0$. By (4.4), we have

$$(4.6) \quad b_n < t \leq b_{n+1} < b_n^\lambda < t^\lambda.$$

On the one hand, it follows from (4.2) and (4.5) that if $n > n_0$, then for every $t \in (b_n, b_{n+1}]$,

$$(4.7) \quad s(t) - \ell = (s(t) - s(b_n)) + (s(b_n) - \ell) \geq -2\varepsilon.$$

On the other hand, it follows from (4.2) and (4.4)–(4.6) that

$$(4.8) \quad s(t) - \ell = (s(t) - s(b_{n+1})) + (s(b_{n+1}) - \ell) \leq 2\varepsilon.$$

Putting together (4.7) and (4.8) gives

$$|s(t) - \ell| \leq 2\varepsilon \quad \text{for every } t \in \bigcup_{n=n_0+1}^{\infty} (b_n, b_{n+1}] = (b_{n_0+1}, \infty).$$

Since $\varepsilon > 0$ is arbitrary, this proves that the ordinary limit of s exists at ∞ and it equals ℓ . \square

Proof of Theorem 2. It is analogous to the proof of Theorem 1. Again, we can show that for every $\varepsilon > 0$ and $\lambda > 1$, there exists an increasing sequence $(b_n : n = 1, 2, \dots)$ of numbers tending to ∞ , while conditions (4.2) and (4.4) are also satisfied.

By the condition (1.9) of slow oscillation, we have

$$(4.9) \quad |s(t) - s(b_n)| \leq \varepsilon \quad \text{whenever} \quad x_0 \leq b_n < t \leq b_n^\lambda$$

(cf. (4.5)). Now, it follows from (4.2), (4.4) and (4.9) that

$$|s(t) - \ell| \leq |s(t) - s(b_n)| + |s(b_n) - \ell| \leq 2\varepsilon$$

$$\text{for every } t \in \bigcup_{n=n_0+1}^{\infty} (b_n, b_{n+1}] = (b_{n_0+1}, \infty).$$

Since $\varepsilon > 0$ is arbitrary, this proves that the ordinary limit of s exists at ∞ and it equals ℓ . \square

Proof of Theorem 3. It hinges on Lemma 3 and Theorem 1.

First, we prove that if the condition (1.7) of slow decrease is satisfied for a single $\varepsilon > 0$, say $\varepsilon := 1$, then we have

$$(4.10) \quad \liminf_{x \rightarrow \infty} \frac{s(x)}{x} \geq 0.$$

Indeed, from (1.7) with $\varepsilon = 1$ it follows that for $p = 1, 2, \dots$ we have

$$s(x_0^{\lambda^p}) - s(x_0) \geq -p, \quad \text{where } x_0 := x_0(1) > 1 \text{ and } \lambda = \lambda(1) > 1;$$

whence we conclude that

$$\frac{s(x_0^{\lambda^p})}{x_0^{\lambda^p}} \geq \frac{s(x_0)}{x_0^{\lambda^p}} - \frac{p}{x_0^{\lambda^p}} \rightarrow 0 \quad \text{as } p \rightarrow \infty.$$

Now, the fulfillment of (4.10) is obvious.

Second, we prove that if the real-valued function $s \in L_{\text{loc}}^1[1, \infty)$ is slowly decreasing, then so is its logarithmic mean $\tau(t)$ defined in (1.4). To this effect, let some $0 < \varepsilon < 1$ be given, and let $x_0 \leq x < t \leq x^\lambda$, where $x_0 = x_0(\varepsilon)$ and $\lambda = \lambda(\varepsilon)$ occur in (1.7) and this time λ is chosen so close to 1 that

$$(4.11) \quad 1 < \lambda \leq 1 + \frac{\varepsilon}{\max\{1, B_2\}},$$

where B_2 is from (3.10).

By definition (1.4), we estimate as follows

$$\begin{aligned}
 (4.12) \quad \tau(t) - \tau(x) &:= \frac{1}{\log t} \int_1^t \frac{s(u)}{u} du - \frac{1}{\log x} \int_1^x \frac{s(u)}{u} du \\
 &= -\left(\frac{1}{\log x} - \frac{1}{\log t}\right) \int_1^x \frac{s(u)}{u} du + \frac{1}{\log t} \int_x^t \frac{s(u)}{u} du \\
 &= \left(\frac{1}{\log x} - \frac{1}{\log t}\right) \int_1^x \frac{s(x) - s(u)}{u} du + \frac{1}{\log t} \int_x^t \frac{s(u) - s(x)}{u} du \\
 &= \left(\frac{1}{\log x} - \frac{1}{\log t}\right) \left\{ \int_1^{x_0} + \int_{x_0}^x \right\} \frac{s(x) - s(u)}{u} du + \frac{1}{\log t} \int_x^t \frac{s(u) - s(x)}{u} du \\
 &= \left(\frac{1}{\log x} - \frac{1}{\log t}\right) (\log x_0) s(x) - \left(\frac{1}{\log x} - \frac{1}{\log t}\right) \int_1^{x_0} \frac{s(u)}{u} du \\
 &\quad + \left(\frac{1}{\log x} - \frac{1}{\log t}\right) \int_{x_0}^x \frac{s(x) - s(u)}{u} du + \frac{1}{\log t} \int_x^t \frac{s(u) - s(x)}{u} du \\
 &=: J_1 + J_2 + J_3 + J_4, \quad \text{say.}
 \end{aligned}$$

It follows from (4.10) that

$$(4.13) \quad \liminf_{x \rightarrow \infty} J_1 \geq 0.$$

Since $s \in L_{\text{loc}}^1[1, \infty)$, we have

$$(4.14) \quad \lim_{x \rightarrow \infty} J_2 = 0.$$

It follows from $x < t \leq x^\lambda$ that

$$(4.15) \quad \frac{1}{\lambda} \leq \frac{\log x}{\log t}.$$

Using inequalities (4.11) and (4.13), by Lemma 3 we estimate as follows

$$(4.16) \quad J_3 \geq -B_2 \left(1 - \frac{\log x}{\log t}\right) \geq -B_2 \left(1 - \frac{1}{\lambda}\right) \geq -B_2(\lambda - 1) = -\varepsilon.$$

Using again the condition (1.7) of slow decrease; (4.11) and (4.15) gives

$$(4.17) \quad J_4 \geq -\frac{1}{\log t} \int_x^t \frac{du}{u} = -\left(1 - \frac{\log x}{\log t}\right) \geq -\left(1 - \frac{1}{\lambda}\right) \geq -(\lambda - 1) \geq -\varepsilon.$$

Combining (4.12)–(4.17) yields

$$\tau(t) - \tau(x) \geq -4\varepsilon \quad \text{whenever } x_0 \leq x < t \leq x^\lambda,$$

provided that x is large enough, where we also took into account the limit relations in (4.13) and (4.14). This proves that $\tau(t)$ is also slowly decreasing. Therefore, we may apply Theorem 1 in Section 2, according to which $\tau(t)$ converges in the ordinary sense as $t \rightarrow \infty$ to the same limit ℓ .

Finally, we apply Theorem A in Section 1 to conclude the ordinary convergence of $s(x)$ to ℓ as $x \rightarrow \infty$, again due to the slow decrease of the function s .

The proof of Theorem 3 is complete. \square

Proof of Theorem 4. It is analogous to the proof of Theorem 3, while using Lemma 4 instead of Lemma 3, and applying Theorem B instead of Theorem A in the last step in the proof. The details are left to the reader. \square

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